# Autoregressive Conditional Density

Higher Moment Risk

Lindberg

# Introduction

Risk management is important.

Market movements are difficult to foresee and rare. They also impact a lot of people.

Volatility—the second moment—is one measure of downside risk but higher moments such as skewness and kurtosis—the third and fourth moment—are of importance as well.

# Summary

This thesis is a model comparison.

The models are GARCH and ACD.

They are compared based on their out-of-sample performance with respect to downside risk.

Conclusion: ACD is better.

# Outline

- Theory
  - Definitions and Models
  - Value at Risk
  - Forecast Comparison
  - Tests
- Analysis
  - Research question
  - Data & Method
  - Summary of Tests
  - Why does it win?
  - Conclusion
  - Further Research

# Theory

# Returns

Let $P_t$ be the price of an asset at time $t$.

The return is given by

$$r_t = \ln\left(\frac{P_t}{P_{t-1}}\right)$$

# Distributions

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The PDF of Normal Inverse Gaussian (NIG) distribution is

$$\frac{\alpha\delta K_1\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\pi\sqrt{\delta^2+(x-\mu)^2}} \exp\left[\delta\gamma+\beta(x-\mu)\right]$$

where

$K_1$ denotes a modified Bessel function of kind three.

The parameters are $\mu$ location $\alpha$ tail heaviness, $\delta$ scale parameter, $\beta$ asymmetry parameter, and $\gamma = \sqrt{\alpha^2-\beta^2}$.

# GARCH model

ARMA(1,1)-GARCH(1,1)-NIG

$$r_t = \mu_t + \varepsilon_t = \mu_t + \sigma_t z_t$$
$$z_t \sim NIG\,(0, 1, \rho, \zeta)$$
$$\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

For $\mu_t$ we use an ARMA(1,1).

GARCH is the go to model.

# ACD model

ARMA(1,1)-ACD(1,1)-NIG

$$
\begin{aligned}
r_t &= \mu_t + \varepsilon_t = \mu_t + \sigma_t z_t \\
z_t &\sim NIG\left(0, 1, \rho_t, \zeta_t\right) \\
\sigma_t^2 &= \omega + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \\
\Psi\left(\rho_t\right) &= f\left(\bar{\rho}_t\right) \\
\Psi\left(\zeta_t\right) &= g\left(\bar{\zeta}_t\right)
\end{aligned}
$$

For $\mu_t$ we use an ARMA(1,1).

# GARCH vs ACD

GARCH is Autoregressive Conditional *Heteroskedasticity* vs

ACD is Autoregressive Conditional *Density*

$(\mu_t, \sigma_t, \rho, \zeta)$ vs $(\mu_t, \sigma_t, \rho_t, \zeta_t)$

So ACD has time varying higher moments.

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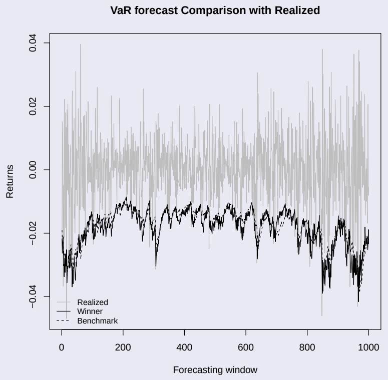
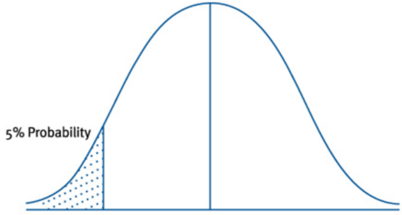
So ACD has time varying higher moments.

- Both models forecast density of returns distribtion at time $t$
  - one-day ahead forecasts
  - given information up until time $t$.
- Want to compare them.
  - As a way of doing this, apply them to tail risk.
  - Why tail risk? Becuase one important application of GARCH is in risk management so if ACD beat GARCH here then it's one piece of evidence that ACD is better than GARCH.

# Value at Risk

## VaR Plot




VaR is a quantile in the returns distribution.

If $r_t < \mathrm{VaR}_t$ a violation occured.

# Forecast Comparison

Models produce forecasts of the parameters that describe the distribution, hence we forecast the density.

VaR is a quantile in this density. We use 1% and 5% quantile.

How do we compare models? How do we know if they are good? Answer: We backtest their density forecast (and violations).

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Three tests

1. Conditional Coverage test (CC-test)
2. VaR Duration test (VaRDur-test)
3. Berkowtiz tail test (Berkowitz-test)

1 and 2 are based on violations. 3 is based on density forecast.

# Test Summary

1. CC-test: correct number of violations and independent.
2. VaRDur-test: violations are memoryless.
3. Berkowitz-test: the shape of the forecasted tail of the density is the same as the observed tail.

Following slides will provide details.

# i) CC-test

$H_0$ : correct number of violations **and** independent.

Expected number of violations $= a \cdot w_F = 50$. Test the null markov chain with a LR test.

Want to accept the null, so the higher the p-value the better.

# ii) VaRdur-test

$H_0$ : violations are memoryless.

Test if the violations sequence follows a Weibull(b=1) distribution, using a LR test.

Want to accept the null, so the higher the p-value the better.

# iii) Berkowitz-test

> *"Loosely speaking, the shape of the forecasted tail of the density is compared to the observed tail."*
> *Berkowitz* [1]

Tested using a LR test.

$H_0$ : Data is independent and N(0,1) i.e. correctly forecasted.

[1] Section 2.2 in Berkowitz, J. 2001, Testing density forecasts, with applications to risk management, *Journal of Business and Economic Statistics*.

# Analysis

# Research question

**Background**

The GARCH model is heavily used in the financial literature as well as in the practice of financial institutions.

The ACD model extend the GARCH model by allowing for a time varying skewness and shape.

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The ACD model extend the GARCH model by allowing for a time varying skewness and shape.

**Question**

*In this thesis we compare the predictive abilities — according to measures of downside risk — of the GARCH and ACD models in an out-of-sample setting.*

# Data & Method

Canadian stock returns from

5066 number of observations. 2000 in $w_E$ and 3066 in $w_F$.

1. Fit a model to the data in $w_E$. [2]
2. Create $w_F$ number of density forecasts, with a rolling window.
3. Save violations and tail density.
4. Backtest using three tests.
5. Put p-values of the test in a table.
6. Look at three tables. Draw conclusions.

[2] We use both 1% and 5% and two models so there will be ACD(0.01) ACD(0.05) and GARCH(0.01) GARCH(0.05).

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Following three slides will have table from each of the test.

The slide after these will summarize these three tables into one table.

[2] We use both 1% and 5% and two models so there will be ACD(0.01) ACD(0.05) and GARCH(0.01) GARCH(0.05).

# i) CC-test

| | expected | actual | cc.LRp | cc.Decision |
|---|---|---|---|---|
| ACD(0.01) | 30 | 30 | | |
| GARCH(0.01) | 30 | 30 | | |
| ACD(0.05) | 153 | 189 | 0.0029 | Reject H0 |
| GARCH(0.05) | 153 | 190 | 0.0015 | Reject H0 |

Table 1:CC-test null hypothesis: VaR violations are correct and independent. High p-value in column $cc.LRp$ is good.

# ii) VaRdur-test

| | b | uLL | rLL | LRp | Decision |
|---|---|---|---|---|---|
| ACD(0.01) | 1.07 | -164.1 | -164.2 | 0.6839 | Fail to Reject H0 |
| GARCH(0.01) | 1.15 | -163.8 | -164.2 | 0.3835 | Fail to Reject H0 |
| ACD(0.05) | 0.93 | -711.7 | -712.8 | 0.1515 | Fail to Reject H0 |
| GARCH(0.05) | 0.93 | -714.6 | -715.6 | 0.1718 | Fail to Reject H0 |

Table 2:VaR Duration Test null hypothesis: VaR violations lack memory. Higher p-value is better. Column $b$ is the Weibull parameter.

# iii) Berkowitz-test

| | uLL | rLL | LRp | Decision |
|---|---|---|---|---|
| ACD(0.01) | -154.2 | -156.7 | 0.08403 | fail to reject H0 |
| GARCH(0.01) | -153.5 | -156.2 | 0.06325 | fail to reject H0 |
| ACD(0.05) | -702.4 | -709.6 | 0.00074 | reject H0 |
| GARCH(0.05) | -702.3 | -710.2 | 0.00037 | reject H0 |

Table 3:Berkowitz tail test null hypothesis: Data is N(0,1). High p-value is better.

# Summary of Tests

| Test Performed | Test Null Hypothesis | a | Model Winner |
|---|---|---|---|
| VaR-test | Viol. are correct and indep. | 0.01 | None |
| VaR-test | Viol. are correct and indep. | 0.05 | None |
| VaRdur-test | Viol. lack memory i.e. Wei(1) | 0.01 | ACD |
| VaRdur-test | Viol. lack memory i.e. Wei(1) | 0.05 | None |
| Berkowits-test | Is N(0,1) i.e. Tail is correct. | 0.01 | ACD |
| Berkowits-test | Is N(0,1) i.e. Tail is correct. | 0.05 | None |

Table 4:According to column *Model Winner* which is a qualitative statement based on comparing the p-value from the test, ACD model is better than GARCH model for $2/6$ test.

# Why does it win?

Because higher moments is important for risks.

Skew and shape measure affect the tail (extreme events) and ACD is more sophisticated (see model specifications).

# Conclusion

Both models are good.

ACD is better, according to higher p-values, at least for our data set and w.r.t our tests.

# Further Research

Do 5 and 10 day ahead forecasts, as well as backtesting it. Because 10 day VaR is used for banks. And because we saw in the bootstrap plot there is difficulty in forecasting 10 days ahead.

Have other assets or even other asset classes. Perhaps their return series behave differently, due to economic reasons, and this will affect the ACD and GARCH model.

Change the model family from standard GARCH to other models such as E-, GJR-, AV-, NAGARCH or apARCH. These models account for assymetric effect (also known as the leverage effect) observed in financial data. The leverage effect arise due to the fact that negative news impact the volatility more than positive news, and this is ignored by the standard GARCH model as seen by its symmetric news impact curve.

Thank you!